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Three-Dimensional Space
Amidst Polyhedral Obstacles

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# On the Piano Movers' Problem: V. The Case of a Rod Moving in Three-dimensional Space Amidst Polyhedral Obstacles. (1)

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*ABSTRACT*

This paper, a fifth in a series, solves some additional 3-D special cases of the 'piano movers' problem, which arises in robotics. The main problem solved in this paper is that of planning the motion of a rod moving amidst polyhedral obstacles. We present polynomial-time motion-planning algorithms for this case, using the connectivity-graph technique described in the preceding papers. We also study certain more general polyhedral problems, which arise in the motion planning problem considered here but have application to other similar problems. Application of these technique to the problem of planning the motion of a general polyhedral body moving in 3-space amidst polyhedral obstacles is also described.

## 1. Introduction.

The strategy used in the analysis of the piano movers' problem begun in [SS1] and continued in [SS3] can be regarded as an optimization of the general but catastrophically inefficient algorithm described in [SS2]; specifically we optimize by treating 'easy' dimensions in a special, direct manner. This strategy can be described in the following general terms. We are given an $n$-dimensional algebraic manifold $FP$, representing the free positions of a body (or group of related bodies) $B$ constrained to move freely in a two- or three-dimensional space bounded by certain obstacles. Our task is to find the connected components of $FP$, or, what is much the same thing, to determine for two given positions $p_1, p_2$ of $B$ whether or not they lie in the same (arcwise) connected component of $FP$. As in [SS1], we can proceed by fixing the values of $k$ of the $n$ parameters specifying a

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position in $FP$. Let $X$ denote the value of the parameters thereby fixed, and consider the corresponding 'fibers' $P(X)$ consisting of all allowed values of the remaining $n-k$ parameters (call them $Y$). In the special case analysed in [SS1], namely that of a rigid polygonal body moving in 2-space amidst polygonal barriers, it is seen that unless $X$ lies in a certain 1-dimensional 'critical' curve of the 2-dimensional space over which it varies, each of the connected components of $P(X)$ can be given a discrete combinatorial characterization which remains invariant under a continuous change of $X$, provided that $X$ avoids the aforementioned critical submanifolds. However, as $X$ crosses a critical submanifold $\beta$ in 2-space, certain typical changes, generally of one of the following three kinds, will occur:

(i) A component of $P(X)$ can disappear (or appear);

(ii) Two components of $P(X)$ can join together into one component, or conversely one component may split into two disconnected subparts.

(iii) The combinatorial description (or 'marking') of a component of $P(X)$ may change, even though the topological structure of $P(X)$ does not change, and even though the connected components of $P(X)$ move continuously as $\beta$ is crossed.

In the elementary special case studied in [SS1], the critical submanifolds $\beta$ are easily characterized in terms of the geometric parameters of the problem (i.e. the shape of the obstacles and of the body). Each critical curve is shown to be of a lower dimension than that of the 2-dimensional space in which $X$ varies, and "crossing rules" are established for each critical submanifold $\beta$. These rules describe how $P(X)$ changes as $X$ crosses $\beta$ from one side to the other. The crossing rules are seen to remain invariant over any connected subset of $\beta$ not containing any singular point. Once this is established, we find the connected components of the set of all noncritical values of $X$; since $k$ is less than $n$, this is a significantly easier problem than the one with which we began. The number of these components is easily seen to be finite, and they can be identified by the collection of 'critical surface patches' forming their boundary. These observations reduce our original problem to a purely combinatorial one. More specifically, we can construct a 'connectivity graph' $CG$ whose nodes are pairs of the form $[R,S]$, where $R$ is a connected component of the noncritical subpart of the region in which $X$ varies, and where $S$ is the combinatorial "marking" of some connected component of $P(X)$ for any (hence every) $X \in R$. An edge will connect $[R_1,S_1]$ to $[R_2,S_2]$ in $CG$ if $R_1,R_2$ are two noncritical regions having a common boundary $\beta$, and if the $P(X)$ component marked $S_1$ on the $R_1$ side of $\beta$ makes contact with the component marked $S_2$ on the $R_2$ side of $\beta$. Finally, let $[X_1,Y_1]$, $[X_2,Y_2]$ be two given configurations of the body (or bodies). Then we can map $[X_1,Y_1]$ and $[X_2,Y_2]$ to nodes $[R_1,S_1]$ and $[R_2,S_2] \in CG$, where $R_j$ is the noncritical component of $k$-space containing $X_j$, and where $S_j$ is the marking of the component of $P(X_j)$ containing $Y_j$, for $j=1,2$. Our original problem is thereby reduced to a simple combinatorial search in $CG$ to check whether $[R_1,S_1]$ and $[R_2,S_2]$ belong to the same component of the finite graph

$CG$.

We also note that there is no need to identify the noncritical regions $R$ explicitly. This fact appears clearly in the algorithm presented in [SS1]. Instead of explicitly identifying these regions, we can proceed in the following simpler manner: First identify all 'critical patches' formed by dividing the critical submanifolds into pieces along lines formed by the intersection of the critical submanifolds. With each such patch $\beta$ associate two 'sides' $r(\beta)$ and $l(\beta)$, and determine the crossing applicable to $\beta$. Then, given a side $R_1$ of some critical surface patch and another side $R_2$ of another patch, determine whether these two sides are connected to each other, that is, whether a point $X_1$ on the $R_1$ side of $\beta_1$ (sufficiently near $\beta_1$) can be connected to a point $X_2$ on the $R_2$ side of $\beta_2$ without going through any critical submanifold. Using any geometric procedure able to make this decision, we can build a modified connectivity graph replacing critical regions by sides of critical surface patches, and connecting $[R_1,S_1]$ to $[R_2,S_2]$ if either $S_1 = S_2$, and $R_1$ and $R_2$ can be connected to each other, without crossing any critical submanifold, and also if $R_1$ and $R_2$ are the two opposite sides of the same critical surface patch and the marking $S_1$ would be changed to $S_2$ when we apply the crossing rules applicable for the patch $\beta$.

It is also of significance that our approach is capable of being used recursively. Our initial problem can be posed as follows: Given the Euclidean $n$-space $E^n$, and a collection of critical manifolds of dimension $n-1$ (these manifolds constitute the boundary of the manifold $FP$ of free positions), find an effective procedure that will determine for any two given points $Z_1,Z_2$ whether they belong to the same connected component of $E^n-V$. Our strategy breaks this problem into two subproblems of size $k$ and $n-k$ by projecting onto $E^k$. First we have to identify connected components of $P(X)$ for each $X \in E^k$ and mark them in some discrete manner. Then we need to identify connected components of $E^k-V_k$, where $V_k$ is the collection of critical submanifolds in the $X$-space. If the dimension $k$ is too large, we can repeat this procedure recursively several times, until $k$ reduces to a manageable size (usually 1 or 2). However, each level of application of the strategy outlined complicates the geometry with which we have to deal, so that in practical terms it is not always possible to apply the proposed approach efficiently. In [SS3] the general recursive principle which we have just sketched is illustrated in the solution of the motion-planning problem for three circles moving in 2-space. Note also that as shown in [SS2], the motion-planning problem can be solved in general using a systematic projection/decomposition technique due to Collins; however, this procedure is very inefficient.

In this paper we continue our study of the mover's problem, by attacking a special case of it having a larger number of degrees of freedom than that considered in [SS1]. Specifically, we consider the case of a rigid rod $B$ moving in 3-dimensional space $V$ bounded by polyhedral walls. This motion-planning problem

involves 5 degrees of freedom, namely 3 translational and 2 rotational parameters. Although problems of 6 degrees of freedom have already been attacked in [SS3], the geometry involved in the case of a moving rod is a bit more complicated. We will show that this problem can be handled by the 'projection' approach outlined above. For this, we begin by projecting $FP$ into the 2-space $A$ of all possible orientations of $B$ (we will identify $A$ with the unit 3-sphere). Thus, for each orientation $\theta$ of $B$, $P(\theta)$ consists of all translations $X$ of $B$ for which $B$ remains free of intersection with the walls when given position/orientation $[X,\theta]$. Section 2 shows how to describe and label the components of $P(\theta)$ by studying the simpler case of a single point moving amidst polyhedral walls in 3-space. This simplified problem involves only 3 degrees of freedom, but its solution uses methods which generalize to similar problems in an arbitrary number of degrees of freedom, which involve only linear constraints.

After these 'purely polyhedral' questions have been treated, we return in Section 3 to our original analysis of a rigid rod. There we describe the geometry of the noncritical regions in the $\theta$-space, develop crossing rules for the critical curves which separate these regions, and finally reduce the problem to a purely combinatorial affair involving an appropriate 'connectivity graph'. Since $\theta$-space is only 2-dimensional, the geometric details of its decomposition into noncritical regions are similar to those pertaining to the decomposition described in [SS1].

## 2. Projecting Onto a Purely Polyhedral Subproblem.

Our problem is to plan a continuous motion of a rigid rod $B$ free to move (i.e. translate and rotate) in 3-dimensional space bounded by a finite collection of polyhedra. Let $P$ be a designated endpoint of $B$, and let $Q$ be the other endpoint. In what follows it will be convenient to specify a general position of $B$ by a quintuple $[x,y,z,\phi,\psi]$, where $X = [x,y,z]$ denotes the Cartesian position of $P$, and where the spherical coordinates $\theta = [\phi,\psi]$ represent the orientation of $B$.

We first wish to eliminate the three translational parameters from the problem, and so project $FP$ onto the 2-dimensional $\theta$-space. To do this, let $\theta$ be a fixed orientation of $B$, and let $B_0$ denote the segment occupied by the rod $B$ when it is placed at orientation $\theta$ with $P$ at the origin. If we constrain $B$ to move without changing this orientation, its motions are all purely translational, and so have just 3 degrees of freedom. It is easy to see that the set $P(\theta)$ of all points $X$ to which $P$ can move while $\theta$ is held fixed is the set of all $X$ such that $(B_0 + X) \cap V^c = \emptyset$, where $V^c$ is the wall region, i.e. the complement of the free space $V$. That is $P(\theta) = (V^c - B_0)^c$. (Here + and - denote pointwise vector addition and subtraction respectively.) Assume that $V^c$ can be cut into finitely many convex polyhedra $W_1, \ldots, W_n$ which intersect only at faces common to two such parts. Then we can write

$$P(\theta) = (\bigcup_i (W_i - B_0))^c$$

Let us agree to write $ext(S)$ for the collection of extreme points of the set $S$, and $conv(T)$ for the convex hull of the set of points $T$. Put $EB_0 = ext(B_0) = \{0, L\theta\}$ (where $L$ is the length of $B$), $EW_i = ext(W_i)$. Then it is easy to see that

$$W_i - B_0 = conv(EW_i - EB_0) = conv(EW_i \cup (EW_i - L\theta))$$

and

$$P(\theta) = (\bigcup_i conv(EW_i - EB_0))^c$$

The problem of characterising $P(\theta)$ and its connected components is therefore a special case of the following more general problem.

**Problem:** Given convex polyhedra $K_1, K_2, \ldots, K_m$, all of which are $n$-dimensional and contained in another $n$-dimensional convex polyhedron $K_0$, find all the connected components of $L = K_0 \cap (\bigcup_i K_i)^c$ and label them unambiguously.

In the following paragraphs we will describe a recursive procedure to solve this problem for arbitrary dimensions $n$.

To solve the polyhedral problem just stated we can proceed in the following recursive manner. We are given $m$ convex $n$-dimensional polyhedra $K_1, K_2, \ldots, K_m$ (some of which may overlap) in Euclidean $n$-space, such that all are contained in the interior of another convex polyhedron $K_0$. Our aim is to compute, and give discrete labelings to, the connected components of the complement $L$ of $\cup K_i$ in $K_0$. We will refer to an $m$-dimensional face of a polyhedron $K$ as an *$m-$face* of $K$; thus faces of maximal dimension are $(n-1)$-faces, and for each $j < n-1$ every $j$-face is a subface of a $(j+1)$-face. An $m$-face will also be called a face of *codimension* $n-m$. By the *interior* of a $k$-face $F$ we mean its interior in the $k$-plane which contains $F$.

The procedure that we are about to present assumes that the polyhedra with which we deal are in 'general position'. To define this property rigorously, we will assume that each of the polyhedra $K_i$ (and $K_0$) are defined as an intersection of a given collection of half-spaces. We then say that our polyhedra are in general position if the following condition holds:

(*) Any collection of $n+1$ distinct bounding half-spaces of the given polyhedra have an empty intersection.

Later in subsequent development of the more general analysis sketched in the introduction, we will say that $\theta$ is a *critical orientation* if the polyhedra $K_i = K_i(\theta)$ appearing for this value of $\theta$ do not satisfy (*). In some cases this violation of (*) may occur because some of the bounding half-spaces happen to coincide with each other, or because such a half-space is about to become redundant, i.e. touches the polyhedron that it bounds at a lower-dimensional face, and so forth. Note therefore that if the polyhedra are in general position then any collection of $n+1$ distinct faces of the given polyhedra have a null intersection, but that this latter condition does not imply condition (*) if degenerate coincidences or redundancies of

the bounding half-spaces do occur.

We prepare for what follows by proving a number of easy auxiliary lemmas.

**Lemma 1:** Suppose that a collection $K_i$ of convex polyhedra is in general position in the sense defined above, and that $F_1, \ldots, F_m$ are a set of faces of these polyhedra, of respective codimensions $c_1, \ldots, c_m$. Then if the interiors of these faces intersect, we have $c_1 + \cdots + c_m \leq n$, and the dimension of the intersection is precisely $n - c_1 - \cdots - c_m$.

Proof: Suppose that the $K_i$'s are in general position, and let $F_1, \ldots, F_m$ have codimensions $c_1, \ldots, c_m$. Each face $F_j$ is the intersection of the $c_j$ $(n-1)$-faces of the polyhedron $K_i$ on which it lies; thus if $c_1 + \cdots + c_m > n$ we must have $F_1 \cap \cdots \cap F_m = \varnothing$. Hence we need only consider the case $d = n - c_1 - \cdots - c_m \geq 0$. Here, let $G_j$ be the interior of $F_j$ for $j=1, \ldots, m$ and suppose that these interiors intersect. Since the interiors of any two distinct faces lying on the same polyhedron are disjoint from one another, it follows that no two faces $F_j$ lie on the same polyhedron. The intersection $I$ of the $G_j$'s contains a relatively open set in the intersection of all the $c_1 + \cdots + c_m$ hyperplanes (of codimension 1) containing an $(n-1)$-face which contains one of the $G_j$. Hence the dimension of the intersection of the $G_j$ is at least $d$.

Next suppose that the closed convex set $I$ has dimension $d'$ greater than $d$, and let $H$ be the hyperplane of smallest dimension containing $I$. Since the boundary of $I$ is plainly contained in the intersection of $H$ with the union of the boundaries of the $G_j$, and since $H$, being of dimension $d'$ cannot be separated by a union of sets of dimension less than $d'-1$, one of those $G_j$, say for definiteness $G_1$, has an (open) face $G'_1$ which intersects $I$ in a set of dimension at least $d'-1 \geq d$. Thus if we replace $G_1$ by $G'_1$, we raise $c_1$ by 1 and lower the dimension of the intersection by at most 1. This step can be repeated till $c_1 + \cdots + c_m = n+1$, at which point a non-null intersection (at least zero-dimensional) must remain. But this is plainly impossible in view of the remarks made in the preceding paragraph. This proves our lemma. Q.E.D.

**Lemma 2:** Let a set of convex polyhedra $K_i$ be in general position, and let $F$ be an $(n-1)$-face of $K_1$. Then the set of all vertices (i.e. extreme points) of $K_1 \cap K_2$ is the set of all intersections of $m$-faces of $K_1$ with $(n-m)$-faces of $K_2$, $1 \leq m \leq n-1$; and similarly, the set of all vertices of $F \cap K_2$ is the set of all intersections of $m$-faces of $F$ with $(n-m)$-faces of $K_2$, $1 \leq m \leq n-2$.

Proof: Since by Lemma 1 no $(m-1)$-face of $K_1$ intersects any $(n-m)$-face of $K_2$, and no $(n-m-1)$-face of $K_2$ intersects an $m$-face of $K_1$, it follows that the unique point $p$ of intersection of an $m$-face $F_1$ of $K_1$ and an $(n-m)$-face $F_2$ of $K_2$ must be interior to both the intersecting faces. Thus linear coordinates can be established near $p$ in which $K_1$ (resp. $F_1$) appears locally as the set $x_1 \geq 0, \ldots, x_{n-m} \geq 0$ (resp.

$x_1 = 0, \ldots, x_{n-m} = 0$), and $K_2$ (resp. $F_2$) appears locally as $x_{n-m+1} \geq 0, \ldots, x_n \geq 0$ (resp. $x_{n-m+1} = 0, \ldots, x_n = 0$). In these coordinates $p$ appears as $(0, \ldots, 0)$, and $K_1 \cap K_2$ appears locally as $x_i \geq 0$, $i=1, \ldots, n$. This makes it plain that $p$ is an extreme point, i.e. a vertex, of $K_1 \cap K_2$.

Conversely, take any $p \in K_1 \cap K_2$, and suppose that it lies in faces $F_1$, $F_2$ of $K_1$, $K_2$ having (largest possible) codimensions $c_1$, $c_2$ respectively. As noted in the proof of the preceding lemma, $p$ must belong to the intersection of the interiors of $F_1$ and $F_2$. Since the $K_i$'s are in general position, we must have $c_1 + c_2 \leq n$, and the intersection of these interiors is of dimension $n - c_1 - c_2$. If this dimension is not zero, then $p$ lies in the interior of this intersection, and consequently cannot be an extreme point of $K_1 \cap K_2$.

This proves the first assertion of lemma 2; The proof of the second is similar and is left to the reader. Q.E.D.

Now we can describe a procedure for computing and labeling the connected components of the set $L$ of the problem mentioned at the end of the preceding section. The input to this procedure is a collection of polyhedra $K_0, K_1, \ldots, K_m$. It will be convenient to assume that each polyhedron $K_i$ is represented by a (finite) collection of half spaces whose intersection is equal to $K_i$. The output of the procedure is the set of all connected components of $L$. Each such component is represented as the set of its $(n-1)$-faces; each such face is in turn represented as the set of all its $(n-2)$-faces, and so on, till eventually we descend to 0-faces, i.e. points, each represented by its coordinates.

Our procedure uses recursion and proceeds downward through successive dimensions $n$. If $n = 1$ the task is trivial, since $K_0$ is an interval, and $K_1, \ldots, K_m$ are subintervals of $K_0$. In this case, each connected component of $L$ is also an open subinterval of $K_0$, and is represented by its endpoints.

Next suppose that $n > 1$. Fix $i \geq 0$, and let $F$ be an $(n-1)$-dimensional face of $K_i$. Compute $F_j = K_j \cap F$ for each $j \geq 1$, $j \neq i$. $F_j$ is represented by the set $U$ of all the half spaces defining $K_j$ and by all half spaces defining $K_i$ except for the half space $H$ whose boundary plane $M$ contains $F$; $H$ is replaced in $U$ by $M$. Condition (*) ensures that each $F_j$ is either empty or is an $(n-1)$-dimensional convex polyhedral subset of $F$. The following lemma shows that condition (*) is hereditary, in the sense that it also holds for these new sets.

**Lemma 3:** Let $K_i$, $F$ and $F_j$ be as above, and let $G_1, \ldots, G_n$ be $n$ distinct $(n-2)$-dimensional hyperplanes contained in the plane of the face $F$ and drawn through the various $(n-2)$-dimensional faces of the convex subsets $F_j$ of $F$. Then the intersection of the $G_i$'s is empty.

**Proof:** Suppose the contrary, i.e. let $x$ be a point in the intersection of $G_1, \ldots, G_n$. Each $G_i$ is the intersection of the hyperplane $M$ containing $F$ with a hyperplane $M_i$ which defines a face $J_i$ of one of the polyhedra $K_j$, such that $x \in J_i$. By (*) it also

follows that all the $M_i$'s are distinct and that none of them coincides with $M$. But this contradicts (*), because the $n+1$ distinct planes $M, M_1, \ldots, M_n$ intersect at $x$. This establishes our lemma. Q.E.D.

**Lemma 4:** Let $G$ be a connected component of the set

$$F - \bigcup_{j\geq 1, j\neq i} K_j.$$

where $i \geq 0$ and $F$ is an $(n-1)$-dimensional face of $K_i$. Let $p$ be an interior point of $G$. Then there exists a neighborhood $U$ of $p$ whose intersection with the side of $G$ outside $K_i$ either lies outside $K_0$ or is contained in $L$.

**Proof:** If $i=0$, the claim is obvious. Otherwise, $p$ does not belong to any $K_j$, $j\geq 1, j\neq i$, and it follows that there exists a neighborhood $U$ of $t$ which is disjoint from each of these $K_j$'s. If near $p$ the exterior side $E$ of $K_i$ is contained in $K_0$, then it is clear that $U \cap E$ has the property asserted. Q.E.D.

We now proceed by considering all faces $F$ of all the polyhedra $K_0, \ldots, K_m$ in turn and decomposing $F - \bigcup_{j\neq i} K_j$ into its connected components. This gives a set $H$ of connected $(n-1)$-dimensional face components, each of which will have been labeled in the manner explained above.

Next we define a relation $\Xi_0$ on $H$ as follows: Two face components $G, G' \in H$ are related by $\Xi_0$ if they have a common $(n-2)$-dimensional subface. (The representation of connected face components that our algorithm provides makes it easy to check this condition). Finally, we take the reflexive and transitive closure of $\Xi_0$ to obtain an equivalence relation $\Xi_1$ on $H$. The following lemma shows that the equivalence classes of this relation approximate a representation of the connected components of $L$ that we seek:

**Lemma 5:** Let $K_j$, $F_j$, $L$ and $H$ be as above, and suppose as before that the $K_j$ are in general position. Then all the faces belonging to the same connected component of the boundary of a connected component of $L$ are equivalent under $\Xi_1$; conversely, each equivalence class of the equivalence relation $\Xi_1$ consists of all faces belonging to some connected component of the boundary of some connected component of $L$.

**Proof:** Let $S$ be a connected component of $L$, and let $T$ be its closure. Let $E$ be the intersection of $T$ with some face $F$ of the polyhedra $K_i$, and let $G$ be a connected component of $E$. $G$ is plainly a connected subset of

$$F - \bigcup_{j\geq 1, j\neq i} K_j,$$

and is in fact a maximal connected such subset; hence $G \in H$. Conversely, by Lemma 4 each $G \in H$ is contained in the closure of some connected component $S$ of $L$, and by maximality of $G$ it is easy to see that $G$ coincides with a connected component of the intersection of $T$ with the face containing $G$. Hence the set $H$ is

the set of all faces of connected components of $L$.

Next let $G,G'$ be two connected face components such that $[G,G'] \in \Xi_0$; let $I$ be their common $(n-2)$-dimensional subface. Take an internal point $p$ of $I$, and a neighborhood of $p$ not intersecting any other $(n-1)$-faces of $L$. We can plainly introduce coordinates near $p$ which make $G$, $G'$ appear locally as two coordinate half-planes, and $L$ appear locally as the 'wedge' of Euclidean space which these half-planes bound. Note that since all half-spaces bounding the polyhedra $K_i$ are linearly independent (as follows from Lemma 1) no other face passes through $p$, or else $p$ would be a boundary point rather than an interior point of $I$. This makes it clear that $G$ and $G'$ are faces of the same connected component of $L$. Hence, for any pair $[G,G'] \in \Xi_1$, $G$ and $G'$ are faces belonging to the same connected component of the boundary of a connected component of $L$. Conversely, suppose that $G$ and $G' \in H$ are faces belonging to the same connected component of the closure $S$ of a connected component of $L$. Then, since the boundary of $L$ is a piecewise flat manifold of dimension $n-1$, one can find a sequence $G=G_1,G_2,\ldots,G_l=G'$ of faces of $S$ such that for each $j<l$ $G_j$ and $G_{j+1}$ meet at some $(n-2)$-dimensional subface of $S$. Then by definition $[G_j,G_{j+1}] \in \Xi_0$, so that $[G,G'] \in \Xi_1$ by transitivity. Q.E.D.

To complete the procedure for finding and labeling connected components of $L$, it remains to determine which connected boundary components bound the same component of $L$. For this, the following simple technique, similar to that described in [OSS], is available. For each boundary component $\Gamma$ choose any point $x_\Gamma \in \Gamma$. If $\Gamma$ is an interior boundary component of some connected component $C$ of $L$, then any straight ray emerging from $x_\Gamma$ upward in some standard vertical direction, must eventually intersect the exterior boundary of $C$, and before the first such intersection it can intersect only interior boundary components of $C$. We therefore draw such a ray from $x_\Gamma$, find all its points of intersection with boundary components, and order these points along the ray. These points divide the ray into intervals which alternately lie either in $L$ or in its complement. We then equivalence every pair of boundary components which intersect our ray in two endpoints of a ray interval all of whose interior points belong to the open set $L$. We repeat this procedure for all components $\Gamma$. It then follows that the extended equivalence relation $\Xi_2$ of $\Xi_1$ thereby obtained has the property that all the faces of each connected component of $L$ are equivalent under $\Xi_2$, and, conversely, each equivalence class of $\Xi_2$ consists of all faces of some connected component of $L$. This completes the description of our recursive procedure.

Next we must discuss situations for which condition (*) is violated. Each $x$ at which (*) is violated will lie in the intersection of $n+1$ faces of our set of polyhedra. This can be expressed as an intersection of finitely many half-spaces with $n+1$ planes of dimension $n-1$. To test for the existence of such a point $x$ we can in principle simply intersect all possible combinations of $n+1$ face planes, and

then check whether their intersection has a non-null intersection with all the additional half spaces defining faces in their respective planes. Note that the first part of this test can be expressed algebraically by saying that the coefficients $(a,b)$ of the $n+1$ planes (which we write as $ax + b = 0$) are linearly dependent. When we do have $n+1$ intersecting faces, then by grouping these faces according to the respective polyhedra that contain them, we can re-express the fact that the $n+1$ planes considered have a non-null intersection, as follows: There exist $t$ polyhedra $K_1',K_2', \ldots ,K_t'$ among the given polyhedra, and a sequence of positive integers $s_1, \ldots ,s_t$ whose sum is $n+1$, and for each $i \le t$ there exists an $(n-s_i)$-face of $K_i'$, such that all these faces have a nonempty intersection.

The sequence of codimensions $s_1, \ldots ,s_t$ appearing in the preceding sentences defines a useful characteristic of the critical situations that we will need to analyze in what follows. Let us illustrate this classification for dimensions $n = 2$ and 3. First suppose $n = 2$. Since there are only two partitions of 3, only the following two critical two-dimensional configurations are possible:

(i) A corner of some polygon $K$ touches an edge of another polygon $K'$ (this corresponds to the partition $3 = 2+1$).

(ii) Three edges, each of a different polygon, intersect at a point (this corresponds to the partition $3 = 1+1+1$).

When $n = 3$, there are four possible partitions of 4. We can list them, each with the correponding intersection conditions, as follows.

(i) Two edges, each of a different polyhedron, meet at a point (corresponding to $4 = 2+2$).

(ii) A corner of one polyhedron touches a face of another (corresponding to $4 = 3+1$).

(iii) Two faces of different polyhedra and an edge of a third one intersect at a point (corresponding to $4 = 2+1+1$).

(iv) Four faces, each of a different polyhedron, intersect at a point (corresponding to $4 = 1+1+1+1$).

This convenient dimensional classification of critical configurations will reappear below.

### 3. Continuous Variations of Components at Noncritical Orientations and Crossing Rules at Critical Orientations

Next suppose that the purely polyhedral problem studied above arises by projecting from a space of higher dimension, as in the case of a rigid rotating rod discussed in Section 1. In this context, our polyhedra and their extreme points will depend continuously on additional parameters, which for notational convenience we denote simply by $\theta$. In conformity with the case of a rotating rod, we will sometimes refer to $\theta$ as an 'orientation'. The procedure described above assigns a

discrete labeling for each connected component of $P(\theta)$ for orientations $\theta$ at which condition (*) holds. Let us agree to call such $\theta$ *noncritical orientations*, and to call orientations $\theta$ at which condition (*) is violated *critical orientations*. Our next aim is to study the way in which the components of $P(\theta)$ and their labels change as $\theta$ varies in a small neighborhood $U$ of a critical orientation $\theta_0$. We prepare for this by a few additional definitions and lemmas.

In what follows we will say that a convex polyhedron $K(\theta)$ depending on one or more parameters $\theta$ *varies continuously with* $\theta$ if the set of half-spaces bounding $K(\theta)$ can be written as $\{H_1(\theta), \ldots, H_n(\theta)\}$, where all the coefficients of each of the $H_j$ depend continuously on $\theta$.

**Lemma 6:** Let $K_i(\theta)$ be a set of convex polyhedra varying continuously with one or more orientation parameters $\theta$, and suppose that for $\theta = \theta_0$ the polyhedra $K_i(\theta)$ are in general position. Then they remain in general position for all $\theta$ sufficiently near $\theta_0$.

**Proof:** Suppose that $\theta_j \to \theta_0$ and that for all $j$ the polyhedra $K_i(\theta_j)$ are not in general position. Then for each $j$ there exist $n+1$ distinct half-spaces bounding certain of the polyhedra $K_i(\theta_j)$ which intersect. As $\theta_j \to \theta_0$, these half-spaces converge to half-spaces $H_1, \ldots, H_{n+1}$ bounding the various $K_i(\theta_0)$. Then plainly $H_1 \cap \cdots \cap H_{n+1} \neq \emptyset$. Since all these limit faces are distinct, this is a contradiction. Q.E.D.

**Lemma 7:** As in Lemma 6, let the convex polyhedra $K_i(\theta)$ vary continuously with $\theta$, and be in general position for $\theta = \theta_0$. Let $K_i(\theta_0) \cap K_j(\theta_0)$ be non-null. Then $K_i(\theta) \cap K_j(\theta)$ is non-null for all $\theta$ sufficiently near $\theta_0$, and the vertices of $K_i(\theta) \cap K_j(\theta)$ vary continuously and converge to the vertices of $K_i(\theta_0) \cap K_j(\theta_0)$. Similarly, if $F(\theta_0)$ is a face of $K_i(\theta_0)$ and for $\theta$ near $\theta_0$ $F(\theta)$ is the corresponding face of $K_i(\theta)$ (i.e. the vertices of $F(\theta)$ converge to the vertices of $F(\theta_0)$), and if $F(\theta_0) \cap K_j(\theta_0)$ is non-null, then $F(\theta) \cap K_j(\theta)$ is non-null for all $\theta$ sufficiently near $\theta_0$, while its vertices vary continuously and converge to those of $F(\theta_0) \cap K_j(\theta_0)$.

**Proof:** By Lemma 2, the set of vertices of $K_i(\theta) \cap K_j(\theta)$ is the set of all intersections of the interiors of $m$-faces $F$ of $K_i(\theta)$ with $(n-m)$-faces $F'$ of $K_j(\theta)$, for all $m=1, \ldots, n-1$. At each such intersection, the planes containing the two intersecting faces are linearly independent, so that they remain independent in a sufficiently small neighborhood of $\theta$, and hence their point $p$ of intersection is unique and moves continuously with $\theta$. Our assertion follows obviously from this remark. Q.E.D.

**Lemma 8:** As in Lemma 7, let the orientation $\theta_0$ be noncritical. Then for $\theta$ in a sufficiently small neighborhood of $\theta_0$, the vertices and $k$-dimensional faces, $k=1, \ldots, n-1$, of $P(\theta)$ remain disjoint from each other and vary continuously with $\theta$.

Proof: We proceed by induction on the number $m$ of polyhedra $K_j(\theta)$ involved, for which purpose it is convenient to assume that the polyhedron $K_0$ containing $L$ is also allowed to vary continuously with $\theta$. Put $K^{(0)}(\theta) = K_0(\theta)$, and successively put $K^{(j+1)}(\theta) = K^{(j)}(\theta) - K_{j+1}(\theta)$. Then we can assume inductively that the assertion of the present lemma is true for some $j$, and must simply prove that it then remains true for $j+1$. We consider only the step from $j=0$ to $j=1$, which is typical. By Lemma 7, the vertices of $K^{(1)}(\theta)$ are those vertices of $K^{(0)}(\theta)$ which do not lie interior to $K_1(\theta)$, plus those vertices of $K_1(\theta)$ which do not lie exterior to $K^{(0)}(\theta)$, plus all intersections of $a$-faces of $K_1(\theta)$ with $(n-a)$-faces of $K^{(0)}(\theta)$, $a=1, \ldots, n-1$. That these vertices remain distinct and vary continuously as $\theta$ varies follows from Lemma 7.

It can be shown in similar fashion that for each dimension $c$, each face of $K^{(1)}(\theta)$ having codimension $c$ is either the intersection of a $c_1$-codimensional face of $K^{(0)}(\theta)$ with a $c_2$-codimensional face of $K_1(\theta)$, where $c = c_1+c_2$; or is that portion of a $c$-codimensional face of $K^{(0)}(\theta)$ which lies exterior to $K_1(\theta)$; or is that portion of a $c$-codimensional face of $K_1(\theta)$ which lies interior to $K^{(0)}(\theta)$. It is plain that all these sets are disjoint from each other. Moreover, we can easily locate their vertices; for example, to locate the vertices of that portion $G(\theta)$ of a $c$-dimensional face $F$ of $K_1(\theta)$ which lies interior to $K^{(0)}(\theta)$, we take $F$ and all its $c'$-dimensional subfaces, for all $c' \leq c$, and form all their intersections with $(n-c')$-dimensional faces of $K^{(0)}(\theta)$ (see Fig. 1). Lemma 7 plainly implies that these vertices vary continuously with $\theta$; hence so does $G(\theta)$. Q.E.D.

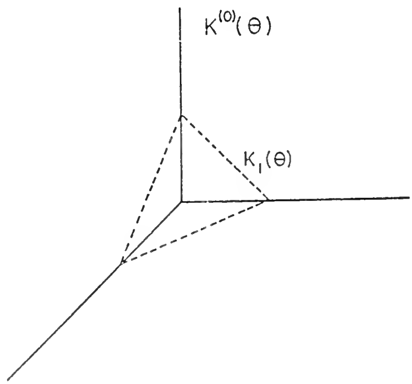



Fig. 1. The vertices of the portion of a 2-face of $K_1(\theta)$ lying interior to a region $K^{(0)}(\theta)$.

**Lemma 9:** Suppose that the polyhedra $K_j(\theta)$ be as above, let the orientation $\theta_0$ be noncritical, and for $\theta$ in a sufficiently small neighborhood $U$ of $\theta_0$ let $C(\theta)$ be the connected component of $P(\theta)$ bearing some fixed combinatorial labeling, which is to say the connected component in whose boundary a particular continuously moving vertex $v(\theta)$ appears. Then the set of continuously moving vertices, edges, etc. lying on the boundary of $C(\theta)$ is invariant for $\theta \in U$, and for $\theta \in U$, all these elements move continuously, as do their interiors and the interior of $C(\theta)$.

**Proof:** Almost everything asserted here has already been proved in Lemma 8; it only remains to show that, for $\theta$ in a small enough neighborhood of each noncritical $\theta_0$, the set of faces, vertices, etc. of each fixed component (identified, say, by identifying some one of its vertices) remains fixed. For this, suppose for the moment that we identify the individual components of $P(\theta)$, for $\theta$ in a sufficiently small neighborhood $N$ of $\theta_0$, by giving some fixed point $Q$ independent of $\theta$ which belongs to each component for all these $\theta$. Now take a vertex $v(\theta)$ and suppose that it is identified by specifying which $n$ planes meet at $v$. Take a spherical neighborhood $U$ of $v(\theta_0)$ small enough so that for $\theta \in N$ no boundary face of $P(\theta)$, other than the three meeting at $v(\theta)$, enter $U$. Take a point $Q'$ interior to $P(\theta_0) \cap U$, and a path in $P(\theta_0)$ from $Q$ to $Q'$. For $\theta \in N$ sufficiently near $\theta_0$, this path lies wholly within the component containing $Q$; and since the faces intersecting at $v(\theta)$ move smoothly and remain linearly independent, their intersection can always be connected to $Q'$ by a straight line segment lying within this same component. This shows that the component to which a given vertex belongs does not change as $\theta$ varies through noncritical positions. Since much the same argument can be given for subfaces of any number of dimensions, our lemma is established. Q.E.D.

Next we consider the phenomena that can occur at critical orientations $\theta_0$, at least for the simplest kind of criticalities. By definition $P(\theta)$ is the complement of the union of various convex polyhedra $K_1(\theta), \ldots, K_r(\theta)$ in some Euclidean space $E^n$. We assume that these polyhedra vary continuously with $\theta$. Since $\theta_0$ is assumed to be critical, there must exist $n+1$ hyperplanes $H_1, \ldots, H_{n+1}$ bounding the polyhedra $K_1(\theta_0), \ldots, K_r(\theta_0)$ which all meet at a common point $X$. Let $N$ denote a sufficiently small convex neighborhood of $X$. Assuming for the moment that $X$ is the only point at which $n+1$ such hyperplanes meet when $\theta = \theta_0$, our problem reduces to that of studying the behavior of the connected components of $V(\theta) = P(\theta) \cap N$ as $\theta$ varies over $U$. (In the following discussion we also assume that no other hyperplane bounding any of the polyhedra passes through $X$; orientations $\theta_0$ at which more than $n+1$ such hyperplanes meet at a point will generally lie on manifolds of codimension at least 2 in $\theta$-space, in which case removal of these points will not affect the connectivity of $G$; see Lemma 1.9 of [SS1]. Hence, if $U$ and $N$ are sufficiently small, it follows that for each $\theta \in U$, $N$ is disjoint from any hyperplane bounding the $K_i(\theta)$ other than the prescribed hyperplanes $H_1, \ldots, H_{n+1}$.)

As before we group the $(n-1)$-dimensional hyperplanes $H_1, \ldots, H_{n+1}$ into collections of hyperplanes, each consisting of hyperplanes bounding the same polyhedron. As previously, we can describe this partitioning by a decomposition of the set $\{1, \ldots, n+1\}$ of integers, and accordingly can write

$$\{1, \ldots, n+1\} = \bigcup_{j=1,\ldots,m} C_j$$

where for each $j$ all hyperplanes $H_i, i \in C_j$, bound the same polyhedron $K_j$. Without loss of generality we can assume that $X = 0$, and can characterize each $H_i$ by the normal unit vector $\alpha_i$ drawn from $H_i$ in the outward direction of the polyhedron $K$ which $H_i$ bounds. Let $H_i^+$ (resp. $H_i^-$) denote the open half-space $\{X : X\cdot\alpha_i > 0\}$ (resp. $\{X : X\cdot\alpha_i < 0\}$); then we can write $V(\theta)$ as

$$V(\theta) = N \bigcap_{j=1,\ldots,m} \left( \bigcup_{i \in C_j} H_i^+ \right) \tag{1}$$

It follows by DeMorgan Laws that $V(\theta)$ can be written as a union of intersections of the form

$$H_{i_1}^+ \cap H_{i_2}^+ \cap \cdots \cap H_{i_m}^+ \cap N,$$

where $i_p \in C_p, p=1, \ldots, m$.

To simplify analysis of the situation before us we assume that the hyperplanes $H_1, \ldots, H_{n+1}$ are such that any $n$ of them are linearly independent at $\theta_0$. By continuity, this property will also hold for all $\theta \in U$ if $U$ is sufficiently small. In this case the behavior of $V(\theta)$ for $\theta \in U$ is essentially determined by the value of $m$. To prove this we first state the following simple lemma:

**Lemma 10:** Let $H_1, \ldots, H_k$ be linearly independent hyperplanes of $E^n$ (with $k \le n$), and let $G_1, \ldots, G_k$ be the closed half-spaces bounded by the respective $H_i$'s. Then the intersection of all the $G_i$'s is a convex cone of $E^n$ having nonempty interior.

**Proof:** The configuration assumed above is linearly isomorphic to that in which the hyperplanes $H_1, \ldots, H_k$ are standard coordinate planes, and since the assertion is plainly true in this transformed configuration, it is also true for the original configuration. Q.E.D.

The fact that any $m$ linearly independent planes, $m \le n$, are linearly isomorphic to standard coordinate planes is used again, implicitly, in the next few paragraphs.

The parameter $m$ appearing in formula (1) can take on a value in any one of the following ranges, for each of which different behavior is observed.

I. Suppose first that $m < n$. Then $V(\theta)$ is connected for each $\theta \in U$. Indeed, consider any two of the intersection sets

$$A = H_{i_1}^+ \cap H_{i_2}^+ \cap \cdots \cap H_{i_m}^+ \cap N$$

and

$$B = H_{j_1}^+ \cap \cdots \cap H_{j_m}^+ \cap N$$

of which $V(\theta)$ is composed. These two sets are contained in the same connected component of $V(\theta)$, because they are linked by the following chain of sets

$$A = W_1 = H_{i_1}^+ \cap H_{i_2}^+ \cap \cdots \cap H_{im}^+ \cap N$$

$$W_2 = H_{j_1}^+ \cap H_{i_2}^+ \cap \cdots \cap H_{i_m}^+ \cap N$$

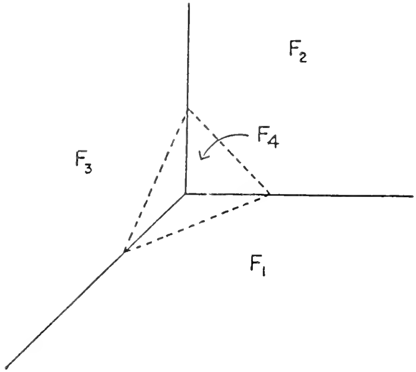
$$\cdots\cdot$$

$$W_m = H_{j_1}^+ \cap H_{j_2}^+ \cap \cdots \cap H_{j_{m-1}}^+ \cap H_{i_m}^+ \cap N$$

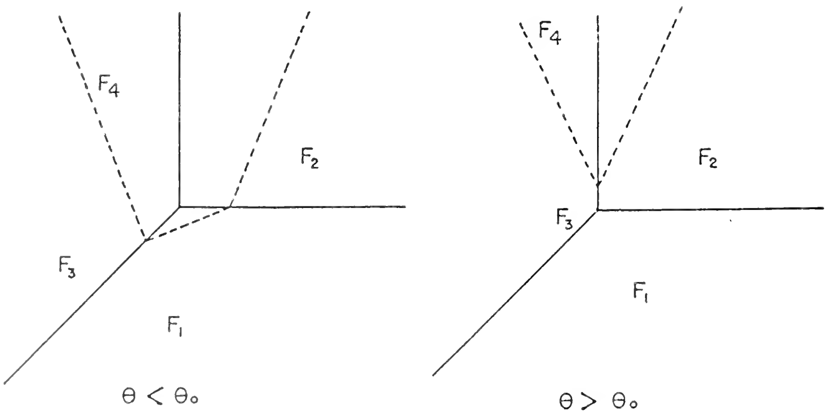
$$B = W_{m+1} = H_{j_1}^+ \cap \cdots \cap H_{j_m}^+ \cap N$$

and moreover each $W_i$ is connected and the intersection of any two successive sets $W_p$ and $W_{p+1}$ is nonempty, by Lemma 1. This shows that in this case $V(\theta)$ is connected for each $\theta \in U$.

II. Next suppose $m = n+1$. Then all the $C_j$'s are singletons, and $V(\theta)$ is the intersection of $n+1$ open half-spaces with $N$. We can take the first $n$ boundary planes of these half spaces, and since they are linearly independent we can establish coordinates in which they have the form $x_i = 0$, $i=1, \ldots, n$, and the half spaces they bound have the form $x_i \geq 0$, $i=1, \ldots, n$. In these coordinates, the $(n+1)$-st plane will have the form $a(\theta) \cdot x \leq b(\theta)$, where without loss of generality we can assume that the coefficient vector has norm $|a(\theta)| = 1$ everywhere in $U$. For $\theta = \theta_0$ all the $n+1$ planes are concurrent at the origin of our coordinates, i.e. $b(\theta_0) = 0$. If any of the components of the vector $a(\theta_0)$ are zero, (say for definiteness that the first component of $a(\theta_0)$ is zero) then $n$ of our $n+1$ planes (specifically $x_2 = 0, \ldots, x_n = 0$, and the $(n+1)$-st plane) are linearly dependent for $\theta = \theta_0$, contrary to assumption. Thus we can assume without loss of generality that all these coefficients are nonzero everywhere in $U$. Hence none of these coefficients change sign in $U$, and we can assume that the first $k$ of them are negative and the rest positive. However, $b(\theta_0) = 0$, so $b(\theta)$ can change sign as it crosses the various sheets of the (algebraic) surface $b(\theta) = 0$ passing through $\theta_0$, which we will call $\Sigma$. If $b(\theta)$ does not change sign when $\Sigma$ is crossed, the geometric configuration within $N$ does not change in any significant way. If $b(\theta)$ changes sign, the geometric change which occurs as $\Sigma$ is crossed depends on the value of $k$. For $k > 0$, all that happens as $b(\theta)$ goes from positive to negative is that the $(n-k)$-face of $V(\theta)$ which lies in the plane of the last $n-k$ $x_j$'s disappears, and that other minor details of $V(\theta)$ change in corresponding ways. However, the connectivity of $V(\theta)$ does not change. But if $k = 0$, then as $b(\theta)$ goes to zero $V(\theta)$ will always shrink to a point (as $\Sigma$ is approached) and then disappear entirely when $b(\theta)$ becomes negative. This is shown in the following figure.




Case (a) As $F_4^+$ move towards the origin, a component shrinks to a point




Case (b) Component boundary changes but component does not vanish

Fig. 2. Two possible crossings of type II.

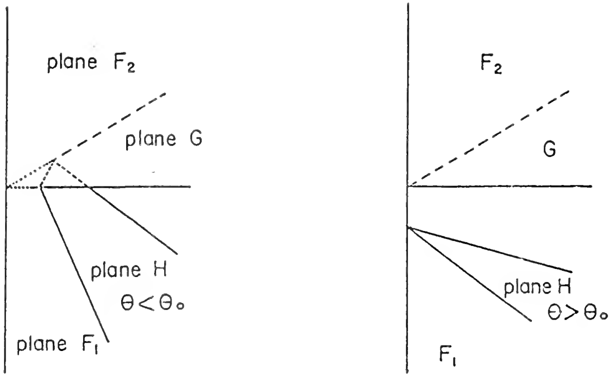
III. Finally suppose that $m = n$. Then all the sets $C_j$ are singletons, with the exception of one doubleton set, say $C_1$. It follows from the preceding discussion and from (1) that $V(\theta)$ is the union of two intersections having the forms

$$G^+ \cap H_2^+ \cap \cdots \cap H_n^+ \cap N,$$

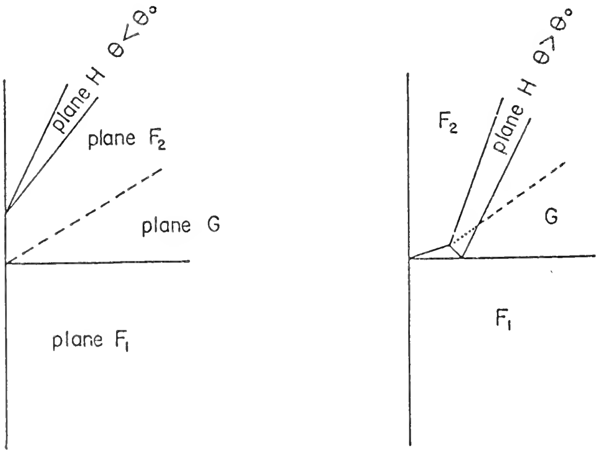
and

$$H^+ \cap H_2^+ \cap \cdots \cap H_n^+ \cap N.$$

It follows from Lemma 1 that each of these sets is nonempty, and will remain nonempty if $\theta$ is sufficiently near $\theta_0$. Hence, without loss of generality we can assume that for $\theta \in U$, $V(\theta)$ is nonempty, and has at most two connected components. In this case one connected component of $P(\theta)$ may split into two components, or vice versa, as $\theta$ passes through $\theta_0 \in U$, though, as the following figure shows, it is also possible that no topological change should occur.




Case (a) Separation of upper octant from lower polyhedron




Case (b) Modification of boundary without component separation

Fig. 3. Two possible crossings of type 3.

These observations show that only three types of *crossing rules* are to be expected in the purely polyhedral case, provided that the critical point being crossed is, so to speak, of a regular criticality. However, the simple analysis which we have given rests upon the assumption that the critical intersecting hyperplanes $H_1, \ldots, H_{n+1}$ are such that any $n$ of them are linearly independent. If this assumption fails to hold, the structure of $V(\theta)$ may become more complicated. As an example illustrating this remark, consider the situation shown in Fig. 4, in which the polyhedral space is assumed to be 5-dimensional. Assume that $F_1, \ldots, F_6$ are faces of the polyhedra in question whose normals $\alpha_1, \ldots, \alpha_6$ span only a two-dimensional subspace of $E^5$. Fig. 4 displays a structure that $V(\theta)$ might have at $\theta_0$, and at two nearby orientations, all drawn in a planar cross-section of $E^5$ spanned by $\alpha_1, \ldots, \alpha_6$. As we see, $V(\theta)$ may have more than two components, which can split and merge in various ways as $\theta$ crosses through $\theta_0$.

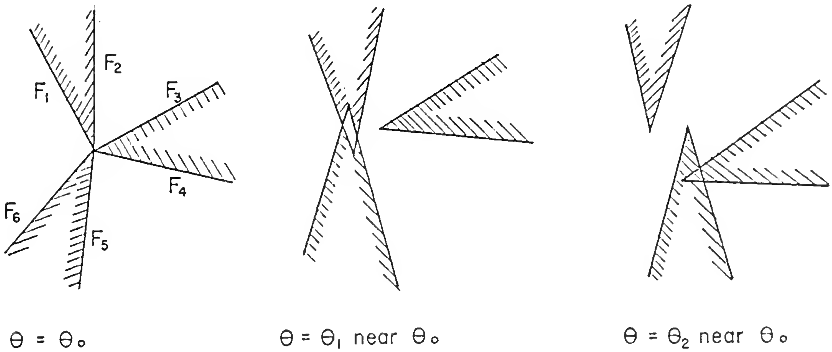



Fig. 4. Crossing a degenerate critical point

However, the following observation implies that the crossing rules in this degenerate case are still similar to those established above.

**Lemma 11:** Let $\theta_0$, $N$, $U$ and $H_1, \ldots, H_{n+1}$ be as above. Let $\theta_i - \theta_0 \in U$, and let $B_i$ be a connected component of $V(\theta_i)$, such that $B_i$ converges (in the Hausdorff metric) to some set $B_0$. Then the interior of $B_0$ is a union of connected components of $V(\theta_0)$.

**Proof:** It is easy to see by continuity that $int(B_0)$ is a subset of $V(\theta_0)$. On the other

hand, each point $z$ on the boundary of $B_0$ (but lying inside $N$) must be a limit point of a sequence of points $Y_t$ which lie on one of the hyperplanes $H_1(\theta_t), \ldots, H_{n+1}(\theta_t)$ for each $t$, so that by continuity $z$ must also lie on one of the hyperplanes $H_1(\theta_0), \ldots, H_{n+1}(\theta_0)$. These simple observations plainly imply our assertion. Q.E.D.

Lemma 11 implies that if $\theta$ approaches $\theta_0$ along a continuous arc consisting of noncritical orientations only, then the limit behavior of the components of $V(\theta)$ at $\theta_0$ will always be a mixture of the crossing rules I-III stated above, that is, some components of $V(\theta)$ may split into several subcomponents; other components of $V(\theta)$ may shrink to a point, while the rest of the components will remain intact and disjoint from each other.

We note that degenerate configurations of the sort that we have forbidden in the main part of our discussion generally correspond to degenerate layouts of some of the walls or of some of the convex subparts of the body $B$. Degeneracies of this kind were analyzed more carefully in [SS1], where a way of treating such configurations systematically is suggested. Specifically, critical surfaces involving these more complex crossing rules can be considered to consist of several layers, infinitesimally close to each other, such that across each of these layers only one standard crossing rule (of the form I-III) applies, so that the final crossing rule is obtained as an appropriate composition of standard rules. A very similar procedure can be devised to handle the case of degenerate purely polyhedral configurations, but we omit the (purely technical) details.

## 4. The Case of a Rotating 3-D Rod.

Having now completed our preparatory work, we return to study the movers' problem for the case of a rigid rod moving in 3-space amidst polyhedral obstacles. As in Section 1, we will assume that the wall region $V^c$ has been cut into (finite) collection $W_1, \ldots, W_n$ of convex polyhedra of 'wall blocks' having disjoint interiors, and intersecting only in 2-dimensional faces common to an intersecting pair $W_i, W_j$ of wall blocks (thus we exclude cases in which two sets $W_i, W_j$ meet only at an edge or at a vertex). As before, let $K_i = K_i(\theta)$ denote the convex polyhedron $conv(ext(W_i) - ext(B_0))$. Let $w_1, \ldots, w_p$ denote all the wall vertices (these are fixed and independent of $\theta$). Note that since some of the $W_i$'s will have coincident faces, there will generally exist polyhedra $K_i$ having coincident faces, i.e. if two wall polyhedra $W_1, W_2$ have a common face $F$, then for each extremity $e \in ext(B_0)$ the polyhedra $K_1$ and $K_2$ can have $F - e$ as a common face. Let us agree to call such common faces *a priori coincident* faces of $K_1$ and $K_2$. Any such a priori coincidence of faces exists independently of $\theta$, and so is an invariant feature of our problem. In the presence of such faces condition (*) will fail *a priori* to hold for the collection $\{K_i\}$ of polyhedra. However, the a priori coincident faces can be ignored in the analysis of Section 2, since they can never form any part of the

boundary of the complement of the polyhedra $K_i$. The presence of a priori coincidences requires only obvious changes in the arguments used in the preceding section and with obvious rewordings they remain valid, so that connected components of $P(\theta)$ can still be found and labeled using the recursive procedure given in Section 2.

Note also that redundancies of hyperplanes bounding the polyhedra $K_i(\theta)$ occur when a face of such a polyhedron shrinks to a face of lower dimension. This however occurs only when two distinct vertices of the displaced polyhedra come together, a situation which occurs only at orientations at which two corners of the rod touch two wall corners. Hence if we rule out degenerate wall configurations at which this can happen, we conclude that at a critical orientation either four faces meet at a point, or two faces simultaneously intersect and become parallel. For this latter kind of criticality, we will also assume that no two wall faces are parallel to one another.

The following revised definition indicates what reformulation of the concept 'critical orientation' is necessary.

**Definition** (a) An orientation $\theta$ is called *noncritical* if the following modified version of (*) holds:

(**) No four faces of the polyhedra $\{K_i(\theta)\}$, no two of which are a priori overlapping faces, have a non-empty intersection, and no two faces which are not a priori coincident simultaneously intersect and become parallel.

(b) An orientation $\theta$ for which condition (**) is violated is called a *critical* orientation.

**More on the Geometry of Critical Orientations**

The crude but useful classification of critical orientations $\theta$ introduced at the end of the preceding section can now be repeated, and separates such orientations into the five following categories:

**Type I critical orientation:** These are orientations $\theta$ for which a vertex of one polyhedron $K_i(\theta)$ lies on a face of another such polyhedron.

**Type II critical orientation:** These are orientations $\theta$ for which an edge of one polyhedron $K_i(\theta)$ meets an edge of another such polyhedron.

**Type III critical orientation:** These are orientations $\theta$ for which a face of one polyhedron $K_1$, a face of another $K_2$, and an edge of a third polyhedron $K_3$ meet at a point.

**Type IV critical orientation:** These are orientations $\theta$ for which four faces, each of a different polyhedron, meet at a point.

**Type V critical orientation:** These are orientations $\theta$ for which two faces simultaneously intersect and become parallel.

The following lemma gives a more direct characterization of faces of the polyhedra $K_i(\theta)$.

**Lemma 12** Let $K$ and $K'$ be two convex 3-dimensional polyhedra. Then the boundary of $K - K'$ is a union of 2-dimensional polygons, which we call *subfaces* of $K - K'$, each of which has one of the three following forms

(a) (a face of $K$) - (a vertex of $K'$);

(b) (an edge of $K$) - (an edge of $K'$);

(c) (a vertex of $K$) - (a face of $K'$).

Similarly, each edge of $K - K'$ is either an edge of $K$ minus a vertex of $K'$, or a vertex of $K$ minus an edge of $K'$, and finally each vertex of $K - K'$ is the difference of a vertex of $K$ and a vertex of $K'$.

**Proof:** A straightforward exercise in convex analysis. Note that some of the polygons (a)-(c) may be coplanar, and that they may overlap one another. Q.E.D.

**Remarks:** (1) In the rod case which concerns us, the second polyhedron $K'$ in the lemma is $B_0$ and is therefore a degenerate polyhedron having no (2-dimensional) faces. Thus case (c) cannot arise in this context. Moreover, when $K'$ is a rod, subfaces $S_1$ of $K - K'$ having type (a) can be written as $S_1 = F_1 - V_1$, where $F_1$ is a face of some wall polyhedron $W_1$, and where $V_1$ is either $P = 0$ or $Q = L\theta$. Then $X \in S_1$ means that if $B$ is placed with the point $P$ at $X$ and is given orientation $\theta$, then the vertex $V_1$ will lie on the face $F_1$ of the walls. The reader will find it easy to give a similar geometric interpretation for subfaces of type (b).

(2) As already noted in the proof of Lemma 12, for some orientations $\theta$ two subfaces of some displaced polyhedron may be coplanar. For example, if the direction $\theta$ lies in the plane of some face wall $F$, then the old face $F$, the face $F$ displaced by $L\theta$, and some boundary edge $e$ of $F$ displaced by $B_0$ will all be coplanar subfaces of the displaced polyhedron, and they might also overlap one another. This situation occurs precisely at critical orientations of type V.

It is not hard to show that the converse of the condition (**) appearing in the definition of a critical orientation $\theta$ can be formulated as follows:

(***) There exist four subfaces $S_1, \ldots, S_4$ of the polyhedra $K_i(\theta)$, none of which lies on an a priori coincident face of the $K_i$'s, which meet at a common point $X$.

By considering all possible combinations of four subfaces, each being of one of the types (a) or (b) appearing in Lemma 12, we can show that only the following 8 critical configurations enter into our rod-motion problem, provided that the geometry of the walls is not 'exceptional' relative to the length of the rod, in the manner described at the end of the preceding section.

**Type L1:** orientations $\theta$ for which one endpoint of the ladder $B$ can touch a wall corner, while the other endpoint lies on a wall face.

**Type L2:** orientations $\theta$ for which both endpoints of $B$ can lie on wall edges.

**Type L3:** orientations $\theta$ for which one endpoint of $B$ can touch a wall corner, while some point interior to or at the end of $B$ meets a wall edge.

**Type L4:** orientations $\theta$ for which one endpoint of $B$ can lie on a wall edge, while the other point lies on a wall face, and some point interior to $B$ touches a wall edge.

**Type L5:** orientations $\theta$ for which $B$ can simultaneously touch three wall edges. (Or, equivalently, one endpoint lies on an edge while two other points touch other edges.

**Type L6:** orientations $\theta$ for which $B$ can simultaneously touch two wall edges, while both of its endpoints lie on wall faces.

**Type L7:** orientations $\theta$ for which $B$ can touch a wall corner, while both of its endpoints lie on wall faces.

**Type L8:** orientations $\theta$ which are parallel to some wall face.

That these are the only critical orientations that need concern us can be deduced as follows. Faces of one or more polyhedra $K(\theta)$ can become parallel to one another when the rod touches two parallel edges (which we ignore by assuming that no two edges of our polyhedra are parallel) or when the rod becomes parallel to a wall face (case (L8)). By Lemma 12, every face, edge, and corner of a polyhedron $K(\theta)$ is either an 'old' face, edge or corner (that is, a face, edge or corner of one of the given polyhedral walls) which we write as 'of', 'oe', 'oc' respectively; or is 'new'. A point on a new face either lies on an old face displaced through the length $L$ of the rod (which has orientation $\theta$), which we write as 'df', or at a position of the endpoint of the rod at which the rod would touch an old edge at some point interior to, or at the end of the rod, which we write as 'te'. The possible new edges, written in a similar notation, are 'de' and 'tc'; the possible new corners are just 'dc'.

To form a critical intersection, these various elements must be combined in one of the four possible codimensional patterns $1+1+1+1$, $2+1+1$, $2+2$, and $3+1$ noted above. No intersection can contain two elements bearing the prefix 'o', since this would correspond to a nonexistent singularity of the walls (e.g. two wall faces belonging to different wall polyhedra would have to intersect); similarly, no two elements bearing the prefix 'd' can occur.

Using these observations, and noting symmetries where possible, we see at once that the possible patterns of critical intersection for the codimensional pattern $1+1+1+1$ are:

$(a)$ $\quad$ *of*, *df*, *te*, *te*

$(b)$ $\quad$ *of*, *te*, *te*, *te*

$(c)$ $\quad$ *te*, *te*, *te*, *te*

(Here we have e.g. used the fact that the two endpoints of the rod are symmetric, so that there is no need to distinguish between pattern (b) and the ostensibly different pattern *df*, *te*, *te*, *te*.) Pattern (a) describes a situation in which the two ends of the rod touch two wall faces, while two of its interior points touch edges; this is our critical configuration L6. On the other hand, pattern (b) is a subcase of L5, which (as will be seen just below) is already critical. Pattern (c) can arise only in degenerate wall configurations, in which four distinct edges intersect the same line.

For the codimensional pattern 2+1+1 we have the possibilities

(*d*) *oe*, *df*, *te*

(*e*) *oe*, *te*, *te*

(*f*) *tc*, *of*, *df*

(*g*) *tc*, *of*, *te*

(*h*) *tc*, *te*, *te*

(Here again, we have used the 'o' and 'd' symmetry to limit our enumeration.) Case (d) is L4. Case (e) is equivalent to L5, since given any configuration in which the rod touches three edges we can slide it, without change of orientation, along its own length until one of its endpoint comes to lie along an edge. Case (f) is our critical intersection L7; case (g) is just the subcase of L3 in which the rod has been pushed along its length, without change of orientation, till one of its points touches a wall face. Case (h) will occur only for isolated directions, and hence can be ignored; it is in effect an exceptional case of L5.

Next we consider the codimensional pattern 2+2. Here the possibilities are

(*i*) *oe*, *de*

(*j*) *oe*, *tc*

(*k*) *tc*, *tc*

Possibility (i) is just case L2, and case (j) can be converted into L3 by sliding the rod along its own length till one of its endpoints lies along the edge that it originally touches. Case (k) can clearly occur only for isolated orientations, and thus can safely be ignored.

Finally we must consider the codimensional pattern 3+1. The possibilities here are

(*l*) *oc*, *df*

(*m*) *oc*, *de*

Case (l) is L1, and as already explained case (m) is equivalent to L3.

Thus the list L1 - L8 appearing above exhausts all the critical configurations

that need concern us.

As we shall see below, some crucial geometric details of $P(\theta)$ at a critical orientation $\theta$ will depend on its type. Nevertheless we can make a few general remarks which apply to all eight types of critical curves. To do this, let $\theta$ be a critical orientation, which we can regard as a point $(\theta_1,\theta_2,\theta_3)$ on the unit 3-sphere. Critical orientations of type L8 lie on great circles on the sphere. To handle the other types of critical curves, we write the equations of the planes containing the four subfaces $S_i$ whose nonempty intersection defines the criticality of $\theta$, and assume for simplicity that the length $L$ of the rod is 1. Each of the subfaces $S$ can have one of the three following forms:

(i) $S$ is a subface of type (a) having the form $F - P = F$, where $F$ is a face of one of the wall polyhedra. In this case the coefficients of the plane containing $S$ are constants independent of $\theta$.

(ii) $S$ is a subface of type (a) having the form $F - Q = F - \theta$, where $F$ is as in (1). In this case, the coefficients of the plane containing $S$ are $a$, $b$, $c$, $d - a\theta_1 - b\theta_2 - c\theta_3$, where $a, b, c, d$ are the coefficients of the plane containing the face $F$.

(iii) $S$ is a subface of type (b) having the form $e - B$, where $e$ is one of the wall edges. Let $(a,b,c)$ be a point on $e$ and let $(k,l,m)$ be a vector in the direction of $e$. Then the coefficients of the plane containing $S$ are easily seen to be the modified cross product

$$\alpha = l\theta_3 - m\theta_2, \ \ \beta = m\theta_1 - k\theta_3, \ \ \gamma = k\theta_2 - l\theta_1, \ \ -a\alpha - b\beta - c\gamma$$

Write the equations of the four planes containing $S_j$, $j=1,\ldots,4$ as

$$a_1x+b_1y+c_1z+d_1=0$$

$$a_2x+b_2y+c_2z+d_2=0$$

$$a_3x+b_3y+c_3z+d_3=0$$

$$a_4x+b_4y+c_4z+d_4=0$$

so that these planes will meet at a point (which may lie at infinity) if and only if we have

$$\begin{vmatrix} a_1(\theta) & b_1(\theta) & c_1(\theta) & d_1(\theta) \\ a_2(\theta) & b_2(\theta) & c_2(\theta) & d_2(\theta) \\ a_3(\theta) & b_3(\theta) & c_3(\theta) & d_3(\theta) \\ a_4(\theta) & b_4(\theta) & c_4(\theta) & d_4(\theta) \end{vmatrix} = 0 \qquad (1)$$

To obtain the algebraic form of the critical curves corresponding to each of the eight geometric types of criticality, we have only to specialize condition (1) to each of these types, as follows.

**Type L1:** Here three of the intersecting planes are of type (i) and one of type (ii) (or vice versa). Equation (1) then clearly defines a plane in $\theta$, whose intersection with the unit sphere is a circle.

**Type L2:** Here two of the intersecting planes are of type (i) and two are of type (ii). Again, Equation (1) defines a plane in $\theta$, so that the corresponding critical curve is also a circle.

**Type L3:** Here three of the intersecting planes are of type (i) or of type (ii), and one is of type (iii). It is easy to see geometrically that equation (1) must define a plane in $\theta$ which passes through the origin, so that the corresponding critical curve is a great circle.

**Type L4:** Here two of the intersecting planes are of type (i), one is of type (ii) and one of type (iii). In this case Equation (1) defines a quadric surface in $\theta$, whose intersection with the unit sphere gives a curve which assumes one of the forms discussed in [OSS].

**Type L5:** Here two of the intersecting planes are of type (i) and two are of type (iii). Again, it is easily seen that Equation (1) defines a quadric surface in $\theta$, leading to critical curves as in type L4 above.

**Type L6:** Here one of the intersecting planes is of type (i), one of type (ii) and two of type (iii). In this case Equation (1) defines a cubic surface in $\theta$, and the corresponding critical curve is the intersection of this surface with the unit sphere.

**Type L7:** Critical orientations of this type can be regarded as degenerate variants of L6, in which the two edges of L6 have come together at a point. Hence type L7 critical curves are also intersections of the unit sphere with a cubic surface.

Finally, as already noted, type L8 critical curves are simply great circles.

It follows from the foregoing discussion that the critical orientations lie on finitely many algebraic curves which decompose the set of noncritical orientations into finitely many *noncritical regions*, i.e. connected open regions of noncritical orientations. Let $R$ be such a region. Lemma 1 implies that for each $\theta \in R$ the set $P(\theta)$ has a fixed number of connected components, each of which varies continuously with $\theta$, and our analysis of the polyhedral case assigns each such component a label which does not depend on $\theta$. By $\sigma(R)$ we shall therefore denote the set of all labels of connected components of $P(\theta)$ for any, hence every, $\theta \in R$, and as in our preceding papers on the movers' problem we call this set the *characteristic* of $R$.

Before continuing with the analysis of the structure of $FP$, we first give a few significant details relevant to the specialization of the labeling procedure described in Section 3 to the case of a rod. First, we need to modify the labeling scheme because, since our displaced polyhedra depend continuously on $\theta$, the 0-faces of the complement $P(\theta)$ of the union of the polyhedra $K_i(\theta)$ will generally depend on $\theta$, so that the labeling of section 2, which represents each such vertex by its coordinates, will not be discrete, but depend on the continuous parameter $\theta$. To avoid this difficulty, we can change the labeling scheme simply by changing the

representation of vertices of $P(\theta)$ as follows. Each such vertex $X$ is the unique intersection of three faces $F_j(\theta)$, $j=1,2,3$, and of no other face of $K_i(\theta)$. Moreover, each face (or rather each subface) of the $K_i(\theta)$ is, by lemma 12, the difference of a wall face and a rod vertex or of a wall edge and the rod itself. Thus, each such subface can be discretely labeled in one of the forms $[W,P]$, $[W,Q]$ (designating a difference of a wall face $W$ and an appropriate endpoint of $B$), or $[E,B]$ (designating a difference of a wall edge $E$ and this same $B$). These labels for faces induce obvious discrete labeling for the corners of $P(\theta)$, namely we can label each such corner by the triple of labels of the faces which pass through it. These labels can then be extended to label edges, faces, and finally components of $P(\theta)$ in the manner described in Section 2 (see also the proof of Lemma 9). To bound the complexity of this labeling procedure, recall that it proceeds recursively downward in dimension, and decomposes each face of the polyhedra that it processes into cells, and classifies these cells as being either subsets of one of the given polyhedra, or as bounding the "free" complemented space. Consider the recursive step of the procedure in which 2-dimensional faces are analyzed. Assume that the polyhedra constituting the boundary of $V$ have altogether $O(n)$ vertices, edges and faces. Each face of each of the displaced polyhedra is either a face of a wall polyhedron, or a face of a wall polyhedron displaced by the vector $L\theta$, or the area swept as a wall edge is displaced by the vector $L\theta$. Thus the displaced polyhedra have altogether at most $O(n)$ vertices, edges and faces. Each 2-dimensional face $F$ of a displaced polyhedron can intersect other displaced polyhedra in a convex polygon, and the portion $F_0$ of $F$ bounding the free space is the complement of the union of these polygons. We claim that the number of corners and edges in this portion of $F$, summed over all faces $F$ of the displaced polyhedra, is at most $O(n^3)$, which implies that the maximal number of connected components of $P(\theta)$ for any orientation $\theta$, and also the size of the label of each connected component of $P(\theta)$, is at most $O(n^3)$. Indeed, our claim is obvious, because a corner $X$ of $F_0$ is either a displaced corner, or the intersection of an edge of one of the $K_i(\theta)$ with a face of some other $K_j(\theta)$, or a point at which three faces $F$, $F'$, $F''$ of the displaced polyhedra meet each other, etc. so that there are $O(n^3)$ such corners in all. It seems rather likely that the actual number of components of $P(\theta)$ in the case of a rod is smaller, possibly $O(n^2)$. However, if instead of a rod we consider a general polyedron moving amidat polyhedral walls in 3-space, the example given in Fig. 5 shows that there exists a layout of $n$ wall faces for which there exist orientations $\theta$ such that $P(\theta)$ contains $\Omega(n^3)$ connected components.

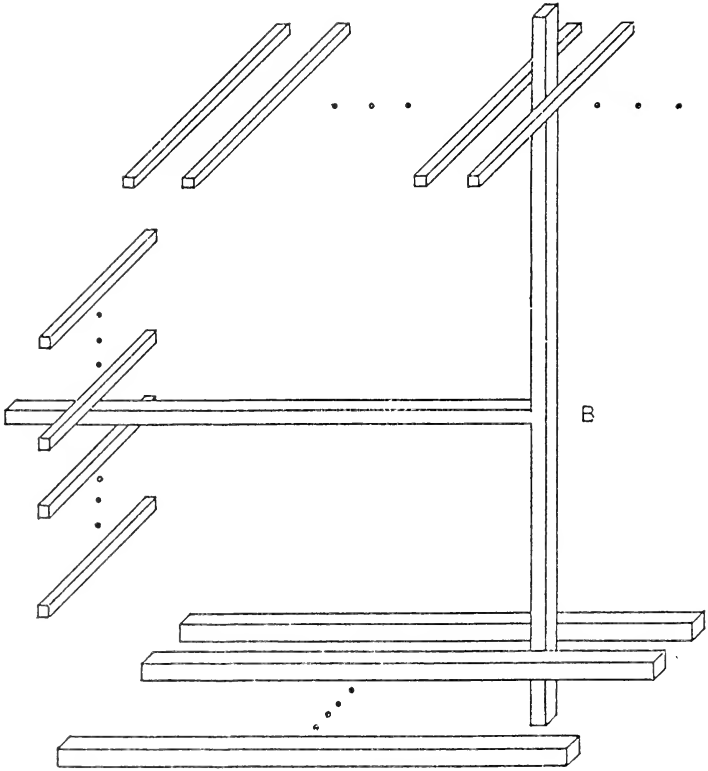



Fig. 5. A wall configuration and an orientation $\theta$ for which $P(\theta)$ has $\Omega(n^3)$ components in the case of a moving polyhedral body.

Note also that if $\theta$ is not a critical orientation, then each corner $C$ of $P(\theta)$ belongs to the boundary of exactly one connected component of $P(\theta)$. Indeed, if $C$ is the intersection point of three faces $F_j(\theta)$, $j=1,2,3$, then the intersection of $P(\theta)$ with a sufficiently small neighborhood of $C$ is contained in the intersection of three open half spaces each bounded by the plane pssing through one of the faces $F_j$, and this intersection is contained in just one component of $P(\theta)$.

The considerations set forth in the preceding paragraphs lead to the following lemmas.

**Definition:** Let $R$ be a noncritical region, and let $L \in \sigma(R)$. For each $\theta \in R$ we let $\phi(\theta,L)$ denote the component of $P(\theta)$ whose label is $L$.

**Lemma 13:** Let $R$ be a noncritical region, and let $L \in \sigma(R)$. Then for each $\theta \in R$ the set $\phi(\theta,L)$ can be expressed as a Boolean combination of nonempty convex sets having the form

$$\phi(\theta,L) = \bigcup_{i_1 j_1, \ldots, i_r j_r} \left( \bigcap_{l=1}^{r} F^{+}_{i_l j_l}(\theta) \right)$$

where $i_1, \ldots, i_r$ range over distinct displaced polyhedra, and where for each $l=1, \ldots, r$, $F_{i_l j_l}(\theta)$ is a face of the $i_l$-th polyhedron $K_{i_l}(\theta)$, and $F^{+}_{i_l j_l}(\theta)$ is the open half-space bounded by the plane passing through $F_{i_l j_l}(\theta)$ and not containing $K_{i_l}(\theta)$.

**Proof:** By definition, for each $\theta \in R$ we have

$$P(\theta) = \left[\bigcup_i K_i(\theta)\right]^c$$

$$= \left[\bigcup_i \bigcap_j F^{-}_{ij}(\theta)\right]^c$$

where $F_{ij}(\theta)$ is the $j$-th face of the $i$-th polyhedron, and where $F^{-}_{ij}(\theta)$ is the closed half-space bounded by the plane passing through $F_{ij}(\theta)$ and containing $K_i(\theta)$. Thus,

$$P(\theta) = \bigcap_i \bigcup_j F^{+}_{ij}(\theta)$$

Using distributivity, we see that $P(\theta)$, and hence each of its connected components $\phi(\theta,L)$, is the union of (nonempty and convex) intersections of the form $\bigcap_{l=1}^{r} F^{+}_{i_l j_l}(\theta)$, as asserted. Q.E.D.

**Lemma 14:** Let $R$ be a noncritical region, and let

$$C(R) = \{[X,\theta] : \theta \in R, X \in P(\theta)\} .$$

Then the connected components of $C(R)$ are the sets

$$C(R,L) = \{[X,\theta] : \theta \in R, X \in \psi(\theta,L)\}$$

where $L \in \sigma(R)$ and where $\psi(\theta,L)$ is defined to be the connected component of $P(\theta)$ having the label $L$.

**Proof:** We first show that each of the sets $C(R,L)$ is connected. Let $[X_1,\theta_1]$, $[X_2,\theta_2] \in C(R,L)$ so that $X_j \in \psi(\theta_j,L)$, $j=1,2$. Let $M_j$ be a corner of $\psi(\theta_j,L)$, i.e. a point where three faces $F_1(\theta_j)$, $F_2(\theta_j)$, $F_3(\theta_j)$ of the displaced polyhedra meet, such that the two points $M_1$ and $M_2$ have the same label (i.e. for each $i=1,\ldots,3$ the faces $F_i(\theta_1)$ and $F_i(\theta_2)$ have the same label.) Let $\alpha(t)$, $t \in [0,1]$ be a continuous path connecting $\theta_1$ and $\theta_2$ in $R$. Then since $\alpha$ does not cross any critical curve, it follows that the three faces $F_i(\alpha(t))$, $i=1,\ldots,3$ vary continuously with $t$, and that the side of $F_i(\alpha(t))$ contained in $\psi(\alpha(t),L)$ remains constant. Let $M(t)$, $e_1(t)$, $e_2(t)$, and $e_3(t)$ denote respectively the point at which the faces $F_i(\alpha(t))$ intersect, and three unit vectors pointing from $M(t)$ along the three edges at which these faces intersect in pairs. Plainly, all these quantities vary continuously with $t$. It follows that if $\delta>0$ is sufficiently small, then the point

$$y(t) = M(t) + \delta(\epsilon_1 e_1(t) + \epsilon_2 e_2(t) + \epsilon_3 e_3(t))$$

is contained in $\psi(\alpha(t),L)$ for all $t \in [0,1]$, for some *fixed* triple of signs $\epsilon_1, \epsilon_2, \epsilon_3$ (e.g. $\epsilon_1 = \epsilon_2 = \epsilon_3 = +1$ if $M_1, M_2$ are convex corners; $\epsilon_1 = \epsilon_2 = \epsilon_3 = -1$ if $M_1, M_2$ are concave corners, etc.). Hence the continuous path $t \to (\alpha(t), y(t))$ connects $[X_1,\theta_1]$ to $[X_2,\theta_2]$ in $C(R,L)$, proving that this set is connected.

To conclude the proof, since the sets $C(R,L)$ are obviously disjoint from each other, it suffices to show that these sets are open. However Lemma 13 implies that $C(R,L)$ can be expressed in the form

$$C(R,L) = \bigcup_{i_1 j_1, \ldots, i_r j_r} \left( \bigcap_{l=1}^{r} \{[X,\theta] : \theta \in R, X \in F^+_{i_l j_l}(\theta)\} \right)$$

and since $F^+_{i_l j_l}(\theta)$ varies continuously with $\theta$, each of the individual sets appearing in the above equation is plainly open in $C(R)$. This completes the proof of the lemma. Q.E.D.

**Lemma 15:** Let $\beta$ be a smooth critical curve section not intersected by other critical curves, let $R$ be a noncritical region bounded by $\beta$, and let $L \in \sigma(R)$. Put $\xi = [R,L]$. Then for each $\theta \in \beta$ and each sequence of orientations $\theta_n \in R$ converging to $\theta$, the sets $\psi(\theta_n,L)$ converge (in the Hausdorff metric of sets) to a unique closed limit set, which we denote by $\phi(\theta,\xi)$, whose interior is contained in $P(\theta)$ and is a union of connected components of $P(\theta)$. Moreover, if $L' \neq L \in \sigma(R)$ and $\eta = [R,L']$, then $int(\phi(\theta,\xi))$ and $\phi(\theta,\eta)$ are disjoint from one another for each $\theta \in \beta$. Thus for each $Z \in V$ the set $\{\theta \in \beta : Z \in int(\phi(\theta,\xi))\}$ is open in $\beta$.

**Proof:** By Lemma 13 we can write

$$\psi(\theta_n,L) = \bigcup_{i_1 j_1, \ldots, i_r j_r} \left( \bigcap_{l=1}^{r} F^+_{i_l j_l}(\theta_n) \right)$$

Each of the half-spaces $F^+_{i_l j_l}(\theta)$ depends continuously on $\theta$ throughout the whole space of admissible orientations. Thus as $\theta_n \to \theta$, the sets $\psi(\theta_n,L)$ converge to a

similar union of intersections of closed half-spaces of the form $\bigcap_{l=1}^{r} closure(F^{+}_{i_l j_l}(\theta))$, and this is the set $\phi(\theta,\xi)$. The interior of this set is equal to the union of the intersections of the corresponding *open* half spaces $F^{+}_{i_l j_l}(\theta)$, which is plainly a subset of $P(\theta)$. Moreover, any point $Z$ in the boundary of $\phi(\theta,\xi)$ is plainly a limit of points $Z_n$ on the boundaries of the sets $\psi(\theta_n,L)$. Since each position $[Z_n,\theta_n]$ is semi-free but not free, so is $[Z,\theta]$. Thus the boundary of $\phi(\theta,\xi)$ is contained in $SFP-FP$, where $SFP$ is the set of all *semi-free configurations* of $B$, that is configurations at which $B$ is either free or touches some wall, but does not penetrate any wall. The set $\phi(\theta,\xi)$ need not be connected, but our last observations imply that $\phi(\theta,\xi)$ contains every connected component of $P(\theta)$ which it intersects.

This proves the first part of the lemma. As to the second part, note that since each of the intersections appearing in the formula displayed above is open, it follows that two such intersections can overlap at $\theta$ iff they overlap for all $\theta' \in R$ sufficiently near $\theta$. Hence two of the terms appearing in the definitions of $int(\phi(\theta,\xi))$ and $\phi(\theta,\eta)$ can overlap only if the connected components to which they belong would be identical for all $\theta'$ near $\theta$. But this would imply $\xi = \eta$, proving that $int(\phi(\theta,\xi))$ and $\phi(\theta,\eta)$ are disjoint if $\xi \neq \eta$. Finally we show that for each $Z \in V$ the set $\beta(Z) = \{\theta \in \beta : Z \in int(\phi(\theta,\xi))\}$ is open in $\beta$. For this, let $\theta$ belong to $\beta(Z)$. By what precedes, $[Z,\theta]$ is a free position of the rod $B$, and hence for all $\theta' \in R$ lying in a sufficiently small neighborhood $U$ of $\theta$, the point $Z$ belongs to some set $\psi(\theta,L')$ with $L'$ *fixed*. (Indeed, choose $U$ to be an arcwise connected neighborhood of $\theta$ in $R$ sufficiently small so that $[Z,\theta'] \in FP$ for all $\theta' \in U$. Then $\{Z\}\times U$ is a connected subset of $C(R)$, and so must be contained in one of its connected components.) Therefore $Z \in \phi(\theta,[R,L'])$, and since we have just shown that $\phi(\theta,[R,L'])$ and $int(\phi(\theta,[R,L]))$ are disjoint if $L' \neq L$, we must have $L' = L$. It therefore follows that $Z \in int(\phi(\theta',\xi))$ for all $\theta' \in \beta$ sufficiently near $\theta$, so that the set $\beta(Z)$ must be open, as asserted. Q.E.D.

**Lemma 16** Suppose that (a portion of) a smooth critical curve $\beta$ separates two connected noncritical regions $R_1$ and $R_2$ and that $R_1 \cup R_2 \cup \beta$ is open. Let $L_1$ (resp. $L_2$) be an element of $\sigma(R_1)$ (resp. $\sigma(R_2)$). Put $\xi_1 = [R_1,L_1]$ and $\xi_2 = [R_2,L_2]$, and let $C_1 = C(\xi_1)$, $C_2 = C(\xi_2)$. Then the following conditions are equivalent:

**Condition A:** There exists a point $\theta \in \beta$ such that the open sets $int(\phi(\theta,\xi_1))$, $int(\phi(\theta,\xi_2))$ have a non-null intersection.

**Condition B:** There exists a smooth path $c(t) = [z(t),\theta(t)]$ in $FP$ which has the following properties:

(i) $c(0) \in C_1$ and $c(1) \in C_2$;

(ii) $\theta(t) \in R_1 \cup R_2 \cup \beta$ for all $0 \le t \le 1$;

(iii) $\theta(t)$ crosses $\beta$ just once, transversally, when $t = t_0$, $0<t_0<1$, and $Z(t)$ is constant for $t$ in the vicinity of $t_0$.

**Proof:** The proof is completely analogous to that of Lemma 1.8 of [SS3], and moreover is general and purely topological. Hence we omit it here.

**Lemma 17:** Let the smooth critical curve $\beta$ separate the two noncritical regions $R_1$ and $R_2$. Let $\beta'$ be a connected open segment of $\beta$ not intersecting any other critical curve, and suppose that $R_1 \cup R_2 \cup \beta'$ is open. Let $L_1$, $L_2$, $\xi_1$, and $\xi_2$ be defined as in Lemma 5. Then the set

$$M = \{\theta \in \beta' : int(\phi(\theta,\xi_1)) \cap int(\phi(\theta,\xi_2)) \neq \varnothing\}$$

is either all of $\beta'$ or is empty.

**Proof:** Since $M$ is the union of all sets of the form

$$\{\theta \in \beta' : Z \in int(\phi(\theta,\xi_1))\} \cap \{\theta \in \beta' : Z \in int(\phi(\theta,\xi_2))\} ,$$

for $Z \in V$, and since by Lemma 15 each of these sets is open, it follows that $M$ is open. Hence we have only to show that $M$ is closed. Suppose the contrary; then there exists $\theta \in \beta'$ such that $int(\phi(\theta,\xi_1))$ and $int(\phi(\theta,\xi_2))$ are disjoint, but for which there also exists a sequence $\theta_n$ of points on $\beta'$ converging to $\theta$ such that for all $n$ the sets $int(\phi(\theta,\xi_1))$ and $int(\phi(\theta,\xi_2))$ intersect each other.

By Lemma 15, for each $n$ the set

$$D_n = int(\phi(\theta_n,\xi_1)) \cap int(\phi(\theta_n,\xi_2))$$

is a union of components of $P(\theta_n)$. Passing to a subsequence if necessary, we can assume that each $D_n$ contains a connected component $C_n$ of $P(\theta_n)$ such that all the $C_n$'s have the same label.

Now since it does not meet any other critical curve other than $\beta$, the subsection $\beta'$ of $\beta$ is characterized by the property that for each $\theta \in \beta'$ four fixed faces $H_j(\theta), j=1,\ldots,4$ of the displaced polyhedra $K_i(\theta)$ meet at a common point $Q(\theta)$ while no four other faces meet. Since each of the connected components $C_n$ must contain more than one corner, we can assume, passing to a subsequence if necessary, that for each $n$ there exists a corner $U_n$ of $C_n$ which does not belong to the intersection of the faces $H_j(\theta_n)$, such that all these corners have the same label, i.e. $U_n$ is the unique point of intersection of three and only three faces $F_1(\theta_n)$, $F_2(\theta_n)$, and $F_3(\theta_n)$ of the polyhedra $K_i(\theta_n)$, which bear fixed designations independent of $\theta = \theta_n$. It is then clear that the sequence $U_n$ converges to some point $U$ as $n \to \infty$.

At least one of the three faces $F_j(\theta_n)$, for specificity say $F_1(\theta_n)$, must be different from any of the four faces $H_j$, and for each $n$ the intersection of $P(\theta_n)$ with a sufficiently small neighborhood of $U_n$ is contained in $C_n$ (i.e. $U_n$ is not a corner common to more than one connected component of $P(\theta_n)$). By continuity, $U \in \bigcap_{j=1}^{3} F_j(\theta)$; moreover, the three faces $F_j(\theta), j=1,2,3$ must still meet at a single

point through which no other plane passes, for otherwise at $\theta$ either $F_1(\theta)$ would be a fifth face passing through $Q(\theta)$ or $\theta$ would lie on the critical curve defined by some other set of four intersecting faces. Thus in either case $\theta$ would have to lie at the intersection of two different critical curves, contrary to our assumption concerning $\beta'$. For large $n$, the three faces $F_1$, $F_2$, $F_3$ will retain their independence and move continuously to the corresponding faces of the polyhedra $K_i(\theta)$ and the interior of the intersection of a small sphere about $U$ with the interior of the region bounded by these faces remains connected. Hence the intersection of $P(\theta_n)$ with any sufficiently small sphere about $U$ is connected for $n$ large and therefore is contained in some fixed component $C_n$ of $P(\theta_n)$.

Hence every open neighborhood of $U$ has a nonempty intersection with $P(\theta)$. Let $N$ be a sufficiently small neighborhood of the origin, and take a point $W \in P(\theta) \cap (U + N)$. Since $[W,\theta] \in FP$ it follows easily that there exist $L'_j \in \sigma(R_j)$, $j=1,2$ such that $W$ belongs to the intersection of the two sets $\phi(\theta,\eta_j)$, where $\eta_j = [R_j,L'_j]$). Thus, by the last assertion of Lemma 15 $W \in int(\phi(\theta_n,\eta_j))$ for all sufficiently large $n$ and for both $j=1$ and $j=2$. But for all sufficiently large $n$ the point $W$ also belongs to $P(\theta_n) \cap (U_n + N)$, and since we have shown that the intersection of $P(\theta_n)$ with every sufficiently small neighborhood of $U$ belongs to $C_n$, $W$ must belong to $C_n$, and thus to $int(\phi(\theta_n,\xi_j))$ for both $j=1$ and $j=2$. Lemma 15 now implies that $\eta_j = \xi_j$, $j=1,2$, so that the sets $int(\phi(\theta,\xi_j))$, $j=1,2$ have a nonempty intersection, contrary to assumption.

This proves that $M$ is also closed, which completes the proof of the lemma. Q.E.D.

Using the lemmas that have now been established, we can proceed with our analysis in a manner completely analogous to that used in our preceding papers (e.g. [SS1], [SS3]). That is, we can define a *connectivity graph* $CG$ whose nodes are the cells $[R,L]$ of $FP$, where $R$ is a noncritical region and where $L \in \sigma(R)$, and whose edges connect adjacent cells in $FP$, i.e. cells $[R_1,L_1]$ and $[R_2,L_2]$ which satisfy the conditions of Lemma 5 at any (hence every) point $\theta$ on the critical curve section $\beta$ separating the adjacent noncritical regions $R_1$ and $R_2$.

The connectivity graph is a finite combinatorial object which models the connectivity of $FP$ exactly. The following main theorem, which is completely analogous to similar theorems of [SS1] and [SS3], makes this point:

**Theorem 1:** There exists a continuous motion $c$ of $B$ through the space $FP$ of free configurations from an initial configuration $[X_1,\theta_1]$ to a final configuration $[X_2,\theta_2]$ if and only if the nodes $[R_1,L_1]$ and $[R_2,L_2]$ of the connectivity graph $CG$ introduced above can be connected by a path in $CG$, where $R_1$, $R_2$ are the noncritical regions containing $\theta_1$, $\theta_2$ respectively, and where $L_1$ (resp. $L_2$) is the label of the connected component of $P(\theta_1)$ (resp. $P(\theta_2)$) containing $X_1$ (resp. $X_2$).

**Remark:** As in [SS1] and [SS3] we assume that $\theta_1$ and $\theta_2$ are not critical

orientations. If either $\theta_1$ or $\theta_2$ lies on a critical curve, we first move $\theta_1$ (or $\theta_2$) into a noncritical region, and then apply the above theorem.

**Proof:** Completely analogous to the proof of Theorem 1.1 of [SS1] and Theorem 1.1 of [SS3], and hence omitted.

The algebraic form of the critical curves and the crossing rules associated with them have been described above. Moreover, the algorithms required for the construction of noncritical regions and the connectivity graph are quite similar in nature to those sketched in [SS1] and [SS3]. For this reason we refrain from giving here additional geometric and algorithmic details concerning the motion-planning technique just outlined.

A crude upper bound on the size of the connectivity graph for an instance of the problem involving $n$ wall faces, edges and vertices, is easily derived, as follows: The number of critical curves is at most $O(n^4)$ (each curve of type L6 is determined by four distinct wall objects chosen out of $n$, which is the worst case). Since all these curves have a bounded degree, the number of possible intersections between them, and thus also the number of noncritical regions, is $O(n^8)$. For each noncritical region $R$, the number of connected components of $P(\theta)$ for any $\theta \in R$ has been shown earlier to be at most $O(n^3)$. Hence the size of the connectivity graph $CG$ is at most $O(n^{11})$, and it is easily seen that $CG$ can be constructed within that time bound.

The motion planning algorithm we have derived is obtained by specializing the polyhedral theory developed in preceding sections to the case of a moving rod in 3-space. The next generalization derivable along these lines is to the case of a general polyhedral body $B$ moving in 3-space amidst polyhedral barriers. Here we must replace the angular orientation $\theta$ by a point parametrizing the 3-dimensional group of rotations in $E^3$. As is well known (see [Ha]) we can parametrize this group by unit quaternions, and hence can think of $\theta$ as belonging to the unit sphere $S^3$ in 4-space. The moving body $B$ can be decomposed into convex polyhedral parts, and then for each $\theta$ the set of free positions of the body becomes the complement of a collection of convex polyhedra of the form $K_l(\theta) = conv(ext(W_i) - ext(B_j(\theta)))$ where $W_i$ designates some one of the convex parts into which the walls have been cut, and $B_j(\theta)$ designates some convex part of the body, rotated by the matrix corresponding to $\theta$. By Lemma 12, the faces, edges, and corners of these $K_l(\theta)$ are as follows:

(a) faces, edges, and corners of the $W_i$, translated by some corner $c$ of $B_j(\theta)$. For purposes of subsequent enumeration, we can designate these geometric elements as $f_C$, $e_C$, and $c_C$ respectively.

(b) edges and corners of the $W_i$, swept respectively into faces and edges of $K_l(\theta)$ by differencing with some edge $e$ of $B_j(\theta)$. We designate these elements as $e_E$ and $c_E$ respectively.

(c) corners of $W_l$ swept to faces by differencing with some face $f$ of $B_j(\theta)$, and designated as $c_F$.

Here the possible critical configurations of four face planes classify as follows. In the codimensional pattern 1+1+1+1, of the four faces entering into the critical intersection, $i$ can have the designation $f_C$ (corresponding to positions in which a body corner lies along a wall face), $j$ have the designation $e_E$ (corresponding to positions in which a body edge touches a wall edge), and $k$ the designation $c_F$ (wall corner on a body face). We must have $i+j+k = 4$, giving $5+4+3+2+1 = 15$ types of singularities. In the codimensional pattern 2+1+1, an edge of one of the forms $e_C$ or $c_E$ combines with two faces. The $e_C$ designates situations in which a corner of the body lies along a wall edge, and $c_E$ describes situations in which a body edge touches a wall corner. This gives $2*(3+2+1) = 12$ more singularity types. Three more types of singularity have the codimensional pattern 2+2, and three have codimensional pattern 3+1. To these we must also add orientations at which a wall edge becomes parallel to a body face, or at which a body edge becomes parallel to a wall face. Counting these as two more types of singular orientations, we obtain altogether $15+12+3+3+2$ or 35 possible types of critical orientations.

If rotations are parametrized by unit quaternions in the manner suggested above, all of these 2-dimensional critical subsurfaces of $S^3$ appear as algebraic surfaces of degree at most eight. Indeed, to apply the rotation $\theta$ to a vector $x$, we simply regard $x$ as a quaternion with zero first component, and form the quaternion product $\theta x \bar{\theta}$, where $\bar{\theta}$ designates the conjugate of $\theta$. This makes it plain that the various geometric coefficients of the rotated $B(\theta)$, and thus also of the displaced polyhedra $K_i(\theta)$, depend only quadratically on $\theta$. Hence concurrency of four of the faces of these polyhedra occurs along the intersection of $S^3$ with an algebraic surface of degree at most eight. Hence, in this case, as in the simpler rod case, we have $O(n^4)$ critical surfaces, but since three of these surfaces can intersect to form corners of the noncritical regions into which they divide $S^3$, there can exist $O(n^{12})$ noncritical regions. In much the same way as in the case of a rod, for any fixed orientation $\theta$ the number of connected components of $P(\theta)$, can be shown to be at most $O(n^3)$. Hence there exists at most $O(n^{15})$ nodes in the connectivity graph.

Lemmas 13, 14, 15, 16, and 17 extend verbatim from the rod case to the more general case presently under consideration. Hence Theorem 1 extends to the case of a moving polyhedron as well. The crossing rules for critical surfaces can be derived from the general principles outlined at Section 3.

However, since in the case of a moving polyhedron the orientation space is three dimensional, its decomposition by critical surfaces into noncritical regions is more complicated than that of the 2-dimensional $\theta$-space in the case of a rod. This can be done using techniques like those described in [OSS] for the

topological analysis of 3-dimensional bodies defined as general semi-algebraic sets. Roughly speaking, one proceeds by computing the intersection curves of all pairs of critical surfaces. Then, for each critical surface $\Sigma$, one considers the collection of all curves of intersection of $\Sigma$ with every other critical surface. These curves partitions the 2-dimensional set $\Sigma$ into connected regions, which can be computed in a way similar to the computation of noncritical regions in the case of a rod. After collecting all these regions on all critical surfaces $\Sigma$ one can compute adjacency relationships between these regions. Two regions are said to be adjacent if they lie on two critical surfaces $\Sigma_1$, $\Sigma_2$, and if their boundaries overlap along a portion of the curve of intersection of $\Sigma_1$ with $\Sigma_2$. From these adjacency relationships one can easily construct connected components of the boundary of each noncritical region, in much the same way as the equivalence relation $\Xi_1$ was constructed in Section 2. To find what boundary components bound the same noncritical region, one can again use the technique described in Section 2, this time extending a great circle on $S^3$ connecting a point on each boundary component with some fixed point on $S^3$, and noting all its pairs of successive intersections with other components, and then equivalencing these pairs of components which must bound the same noncritical region (see [OSS] for more detail).

This completes our brief outline of the way in which the case of a moving polyhedron can be treated. We note that a restricted version of this problem has already been studied in [LPW]. There, however, the moving polyhedron is assumed either not to rotate at all, or rotate only a few times at some discrete positions.

**Remark:** Other generalizations of this 'polyhedral' scheme to more complex systems, such as hinged polyhedral bodies, are also possible. Although such problems can be considered within the general scheme presented in this paper, their analysis becomes substantially more complicated due to the increasing number of non-translational degrees of freedom involved.

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